Down the Rabbit Hole: An Introduction to p-Adic Numbers

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Fun fact: $.99999\ldots = 1$

Proof 1: Let $x = .99999\ldots$

$$10x = 9.99999\ldots = 9 + .99999\ldots = 9 + x.$$

So $9x = 9$, hence $x = 1$.

Fun fact: $.99999\ldots = 1$

Proof 2: Recall the geometric series formula,

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}.$$

Remember that $.99999\ldots$ is a shorthand for

$$.99999\ldots = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots = \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots\right)$$

Applying the geometric series formula, we get

$$\frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots\right) = \frac{9}{10}\left(\frac{1}{1 - \frac{1}{10}}\right) = 1.$$

Bizarre fact: $\ldots 99999 = -1$ (???)

Proof 1: Let $x = \ldots 99999$.

$$\begin{array}{r} \ldots 9\,9\,9\,9\,9 \\ + \qquad\quad 1 \\ \hline \end{array}$$

Bizarre fact: $\ldots 99999 = -1$ (???)

Proof 1: Let $x = \ldots 99999$.

$$\begin{array}{r} 1\,1\,1\,1 \\ \ldots 9\,9\,9\,9\,9 \\ +\qquad\quad 1 \\ \hline \ldots 0\,0\,0\,0\,0 \end{array}$$

So $x + 1 = 0$, hence $x = -1$.

Bizarre fact: $\ldots 99999 = -1$ (???)

Proof 2: Let $x = \ldots 99999$.

$$x = \ldots 99999 = \ldots 99990 + 9 = 10x + 9.$$

So $-9x = 9$, hence $x = -1$.

Bizarre fact: $\ldots 99999 = -1$ (???)

Proof 3: Let's use the geometric series formula.

$$
\begin{aligned}
\ldots 99999 &= 9 + 9 \cdot 10 + 9 \cdot 10^2 + 9 \cdot 10^3 + \ldots \\
&= 9\big(1 + 10 + 10^2 + 10^3 + \ldots\big) \\
&= 9 \left(\frac{1}{1 - 10} \right) \\
&= -1.
\end{aligned}
$$

Bizarre fact: $\ldots 99999 = -1$ (???)

Let's compute $(\ldots 99999)^2$.

$$\begin{array}{r} \ldots 99999 \\ \times \ldots 99999 \\ \hline \end{array}$$

Bizarre fact: $\ldots 99999 = -1$ (???)

Let's compute $(\ldots 99999)^2$.

$$
\begin{array}{r}
\ldots 8\,8\,8\,8 \\
\ldots 9\,9\,9\,9\,9 \\
\times\ \ldots 9\,9\,9\,9\,9 \\
\hline
\ldots 9\,9\,9\,9\,1
\end{array}
$$

Bizarre fact: $\ldots 99999 = -1$ (???)

Let's compute $(\ldots 99999)^2$.

$$
\begin{array}{r}
\ldots 9\,9\,9\,9\,9 \\
\times \ldots 9\,9\,9\,9\,9 \\
\hline
\ldots 9\,9\,9\,9\,1 \\
\ldots 9\,9\,9\,1 \\
\ldots 9\,9\,1 \\
\ldots 9\,1 \\
\ldots 1
\end{array}
$$

Bizarre fact: $\ldots 99999 = -1$ (???)

Let's compute $(\ldots 99999)^2$.

$$
\begin{array}{r}
\ldots 3\,2\,1 \\
\ldots 9\,9\,9\,9\,1 \\
\ldots 9\,9\,9\,1 \\
\ldots 9\,9\,1 \\
\ldots 9\,1 \\
+\ldots 1 \\
\hline
\ldots 0\,0\,0\,0\,1
\end{array}
$$

So $(\ldots 99999)^2 = 1$.

Conundrum!

What's going on here???

Common sense tells us that ...99999 is not a number. . .

. . .but these "proofs" make it feel like there's something there.

Conundrum!

Two options:

1. Pretend like you never saw this. Never think about this again. Repress the memories.

2. Keep an open mind. Embrace the mystery of the unknown. Chase this crazy idea down the rabbit hole.

Power series

Ex: $1 + x + x^2 + x^3 + \dots \quad 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$

Power series = Really really long polynomial

Think of a power series more like a number than like a function: we can do algebra with power series without plugging anything in for x.

Formal power series

We can add and multiply power series, it doesn't matter where they converge!

Addition:

$$(a_0 + a_1x + a_2x^2 + \ldots) + (b_0 + b_1x + b_2x^2 + \ldots) =$$
$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots$$

Multiplication:

$$(a_0 + a_1x + a_2x^2 + \ldots) \cdot (b_0 + b_1x + b_2x^2 + \ldots) =$$
$$(a_0b_0) + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \ldots$$

Formal power series

The geometric series formula holds for formal power series.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

The expression "$\frac{1}{1-x}$" just means an element which when multiplied by $1-x$ gives 1.

$$\begin{aligned}(1-x)(1+x+x^2+x^3+\dots) &= 1+x+x^2+x^3+\dots \\ &\quad -x-x^2-x^3-\dots = 1\end{aligned}$$

Numbers as polynomials

$$\begin{aligned} \text{Integers written in decimal} &\approx \text{polynomials in the “variable” 10} \\ \text{Digits} &\approx \text{coefficients} \\ 3729 &\approx 3 \cdot 10^3 + 7 \cdot 10^2 + 2 \cdot 10 + 9 \end{aligned}$$

Only difference: we have “carries” when we do arithmetic with numbers.

If integers are like polynomials…

…then what are like power series?

10-adic numbers!

10-adic numbers are formal power series in the “variable” 10.

Ex: $2+3\cdot 10+7\cdot 10^2+1\cdot 10^3+\ldots \quad 1+10+2\cdot 10^2+6\cdot 10^3+\ldots$

Only allow “coefficients” in $\{0,1,\ldots,8,9\}$.

Addition and multiplication work just like for power series, except we might have to carry.

Key observation: If a and b are 10-adic numbers, then the coefficient of 10^n in $a+b$ or $a\cdot b$ only depends on the coefficients of 10^k in a and b with $k\le n$.

What about subtraction?

Not sure if

$$-(2+3\cdot 10+7\cdot 10^2+1\cdot 10^3+\ldots)=-2-3\cdot 10-7\cdot 10^2-1\cdot 10^3-\ldots$$

or whether this even makes sense...

$-1 = 9 + 9 \cdot 10 + 9 \cdot 10^2 + \dots$

If a is a 10-adic number, then $-a = (-1) \cdot a$.

$$-1 = 9 + 9 \cdot 10 + 9 \cdot 10^2 + \dots$$

Example: If $a = 2 + 3 \cdot 10 + 7 \cdot 10^2 + \dots$, then

$$\begin{aligned} -a &= (9 + 9 \cdot 10 + 9 \cdot 10^2 + \dots) \cdot (2 + 3 \cdot 10 + 7 \cdot 10^2 + \dots) \\ &= 8 + 6 \cdot 10 + 2 \cdot 10^2 + \dots \end{aligned}$$

Question: Can you find a shortcut for computing the 10-adic expansion of $-m$ for a positive integer m?

b-adic integers $\mathbb{Z}_b$

Every integer has a base b expansion for any $b \geq 2$.

b-adic integers:
$\mathbb{Z}_b = \{a_0 + a_1 b + a_2 b^2 + \ldots : a_k \in \{0, 1, \ldots, b-1\}\}$

We can also write a b-adic integer as a number that goes on forever to the left.

$$\ldots 1984537192 \in \mathbb{Z}_{10}$$
$$\ldots 10101011010 \in \mathbb{Z}_2$$
$$\ldots 3130340214 \in \mathbb{Z}_5$$

Zero divisors

Important property of $\mathbb{Z}$: If $mn = 0$, then $m = 0$ or $n = 0$.

If b is composite, then $\mathbb{Z}_b$ doesn't have this property!

Example: In $\mathbb{Z}_{10}$ we have

$$(\ldots 9879186432)(\ldots 8212890625) = 0.$$

If p is prime, then $\mathbb{Z}_p$ has no zero divisors.

This is one reason people focus on the p-adics.

Let’s tour the strange world of p-adics!

Periodic p-adics

In $\mathbb{Z}_5$ we have

$$\ldots 444444 = -1.$$

Can we figure out the values of other periodic numbers?

$$\ldots 13131313 \stackrel{?}{=}$$

Let $x = \ldots 13131313$. Then

$$\begin{aligned} 10^2 x &= \ldots 13131300 \\ 10^2 x + 13 &= \ldots 13131313 \\ 10^2 x + 13 &= x. \end{aligned}$$

So $x = \frac{13}{1-10^2}$ (written in base 5).

$$x = -\frac{8}{24} = -\frac{1}{3} \quad \text{(base 10)}.$$

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Periodic p-adics

$$\ldots 13131313 = -\frac{1}{3} \quad \text{in } \mathbb{Z}_5$$

There are fractions in $\mathbb{Z}_p$!

Questions:

1. Which fractions are in $\mathbb{Z}_p$?
2. Can you characterize the periodic p-adic numbers?

pth powers

Let $p = 5$.

$$\lim_{n\to\infty} 2^{5^n} \stackrel{?}{=}$$

Our gut reaction might be to say "∞!" or "does not exist!" but in the p-adic world, stranger things can happen.

2^{5^n}	n
2	0
112	1
$\ldots 2220212$	2
$\ldots 0321212$	3
$\ldots 1331212$	4
$\ldots 1431212$	5
$\ldots 2431212$	6
$\ldots 2431212$	7

pth powers

Things get even weirder when replace 2 with other numbers.

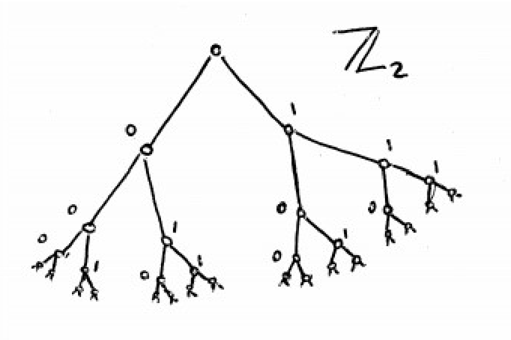
$\lim_{n\to\infty} b^{5^n}$	b
$\ldots 0000001$	1
$\ldots 2431212$	2
$\ldots 2013233$	3
$\ldots 4444444$	4
$\ldots 0000000$	5
$\ldots 0000001$	6
$\ldots 2431212$	7
$\ldots 2013233$	8

$\lim_{n\to\infty} b^{5^n}$	b
$\ldots 4444444$	9
$\ldots 0000000$	10
$\ldots 0000001$	11
$\ldots 2431212$	12
$\ldots 2013233$	13
$\ldots 4444444$	14
$\ldots 0000000$	15
$\ldots 0000001$	16

What's the deal with $\ldots 2431212$ and $\ldots 2013233$?
Hint: $(\ldots 2431212)^2 = (\ldots 2013233)^2 = \ldots 4444444$.

$\mathbb{Z}_p$ is a tree

We can visualize $\mathbb{Z}_p$ as the leaves of a tree. Here's an example when $p = 2$.



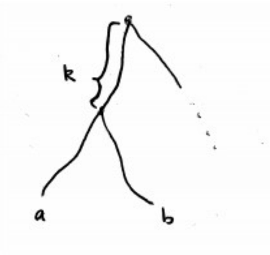
Think of a *p*-adic number as directions for how to get from the root to a leaf.

Distance in $\mathbb{Z}_p$

Define the *distance* $d(a,b)$ between $a,b \in \mathbb{Z}_p$ to be

$$d(a,b) = \frac{1}{p^k}$$

when a and b have the same first k digits/differ for the first time at the $k+1$th digit.



So a and b are "close" if they have a lot of digits in common.
A p-adic number is "small" if it starts with lots of 0's.
Ex: $d(\ldots 232143, \ldots 012143) = \frac{1}{5^4}$

p-adic disks

The *disk* centered at $a \in \mathbb{Z}_p$ with radius $r = \frac{1}{p^k}$ is

$$\begin{aligned} D(a,r) &= \{b \in \mathbb{Z}_p : d(a,b) \leq r\} \\ &= \{b \in \mathbb{Z}_p : \text{ the first } k \text{ digits of } b \text{ agree with } a\} \end{aligned}$$

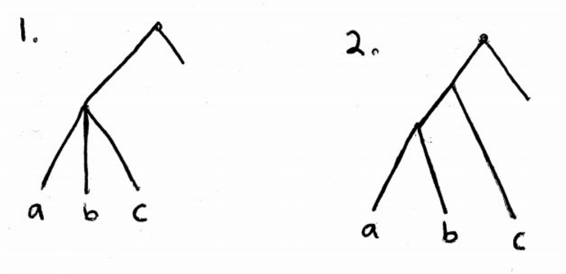
"Every point in a p-adic disk is the center."

"If two disks overlap, then one is contained inside the other."

p-adic triangles

If $a, b, c \in \mathbb{Z}_p$ are three distinct points, we can think of them as forming the vertices of a triangle in $\mathbb{Z}_p$.

Two possibilities:



"Every p-adic triangle is isosceles."

Questions

Here are some questions for you to think about.

1.$(\ldots 1216213)^2 = 2$ in $\mathbb{Z}_7$, so $\sqrt{2} \in \mathbb{Z}_7$! For which primes p is $\sqrt{2} \in \mathbb{Z}_p$?

2. $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}$. Can you find a simple description of $\mathbb{Q}_p$, the field of fractions of $\mathbb{Z}_p$? How does $\mathbb{Q}_p$ fit into the tree picture?

3. The units in $\mathbb{Z}$ are ± 1. Can you describe all the units in $\mathbb{Z}_p$? Hint: there are a lot more.

Thanks!